Exercise 29

(a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = e^{x-1}\,\mathbf{i} + xy\,\mathbf{j}$ and C is given by $r(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}$, $0 \le t \le 1$.

(b) Illustrate part (a) by using a graphing calculator or computer to graph C and the vectors from the vector field corresponding to $t = 0$, $1/\sqrt{2}$, and 1 (as in Figure 13).

Solution

With this parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^1 \left\langle e^{x(t)-1}, x(t)y(t) \right\rangle \cdot \frac{d}{dt}\langle t^2, t^3 \rangle\,dt \\
&= \int_0^1 \left\langle e^{t^2-1}, (t^2)(t^3) \right\rangle \cdot \langle 2t, 3t^2 \rangle\,dt \\
&= \int_0^1 [e^{t^2-1}(2t) + t^5(3t^2)]\,dt \\
&= \int_0^1 (2te^{t^2-1} + 3t^7)\,dt \\
&= \int_0^1 2te^{t^2-1}\,dt + 3\int_0^1 t^7\,dt.
\end{aligned}$$

Make the following substitution in the first integral.

$$\begin{aligned}
u &= t^2 - 1 \\
du &= 2t\,dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{0^2-1}^{1^2-1} e^u\,du + 3\int_0^1 t^7\,dt \\
&= \int_{-1}^0 e^u\,du + 3\int_0^1 t^7\,dt \\
&= (1 - e^{-1}) + 3\left(\frac{1}{8}\right) \\
&= \frac{11}{8} - e^{-1}.
\end{aligned}$$

Below is a plot of the vectors from the vector field corresponding to $t = 0$, $1/\sqrt{2}$, and 1.